Self-overlap as a method of analysis in Ising models

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The damage spreading method (DS) provided a useful tool to obtain analytical results of the thermodynamics and stability of the $2D$ Ising model –amongst many others–, but it suffered both from ambiguities in its results and from large computational costs. In this paper we propose an alternative method, the so called self-overlap method, based on the study of correlation functions measured at subsequent time steps as the system evolves towards its equilibrium. Applying markovian and mean field approximations to a $2D$ Ising system we obtain both analytical and numerical results on the thermodynamics that agree with the expected behavior. We also provide some analytical results on the stability of the system. Since only a single replica of the system needs to be studied, this method would seem to be free from the ambiguities that afflicted DS. It also seems to be numerically more efficient and analytically simpler.

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I. INTRODUCTION

The damage spreading (DS) method [1] is a remarkable tool amongst the many ones developed in recent years in the effort to understand the dynamics of cooperative systems. Very roughly speaking the goal of the method is to study the stability of a cooperative system under a small perturbation: if perturbations die off after some time then the system must be in a stable, ordered state; if small perturbations always get amplified however then the system must be in a disordered, chaotic state. By studying how far away the final states are from the initial ones given the initial perturbation one can get information about the system such as, for example, the Lyapunov exponents.

Of course there is a key aspect that differentiates cooperative systems from classical dynamical systems. Namely, that in the former case given the complexity of the systems under study we almost never have at our disposal a detailed analytical solution to the equations of motion in order to study the system's stability under perturbations. Here is where the DS method comes in: its operational side amounts to an algorithm designed to study how small perturbations spread within the system by working in detail how each of the system's components react to the changes. The method has been applied to many different dynamical systems such as Ising systems [2, 3, 4, 5, 6, 9, 10, 11], Kauffman networks [12, 13, 14], spin glasses [15, 16], cellular automata [17] amongst others, yielding in many cases useful information about their evolution and stability. Succinctly speaking, the algorithm analyzes the evolution of two almost identical states of the system. The damage (difference between the two initial states) is specified as part of the initial conditions. That is, on one side we have a specified state of the system, and on the other we have a replica that only differs in a small perturbation (the damage) from this original state. One then fixes the stochastic evolution to be the same for each replica (in a Monte Carlo simulation the method imposes the same random numbers at each step of time on both copies for instance). As we let the two copies evolve, the method analyzes their distance (Hamming distance) as a function of time. Useful information about the system can then be extracted from this, not only numerically but in some cases also analytically.

However, as was shown in [7], [8] and [9] (and references therein) DS has been shown to be ill-defined in the sense that different –and equally legitimate– algorithmic implementations of the same physical system's dynamics can yield different DS properties. This ambiguity stems from the fact that while the transfer matrix for the evolution of a single system is completely determined by the one-point correlation functions [9], the simultaneous evolution of two replicas however is governed by a joint transfer matrix determined by two-point correlation functions. For example, Glauber and both standard and uncorrelated heat bath (HB) algorithms satisfy detailed balance with respect to the same Hamiltonian. It follows that these three different update rules generate the same equilibrium ensemble and are therefore equally legitimate to mimic the evolution in time of an Ising system coupled to a thermal reservoir. Accordingly, the one-point correlation functions for the three cases coincide and the corresponding transfer matrices for single systems are identical. On the other hand the two-point functions for HB and Glauber dynamics are different; hence damage evolves differently in either case (see [9] and references for a extended quantitative version of this argument). As long as the results depend on the algorithm being implemented, one can not assert that the

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results obtained from a given DS analysis are conclusive and unambiguous.

This handicap is a major motivation in order to search for an alternative method of stability analysis. Our goal in this paper will be to propose a different approach to study the stability of cooperative systems. By relying heavily on the above mentioned fact that the evolution of a single system is determined only by the one-point correlation functions we will try to eliminate some of ambiguities found in the DS method.

In order to be specific, as a test case we will focus on the study of a well-known type of system: Ising models. In [2] Votja tackled the $2D$ Glauber-Ising model via DS. He obtained results on the thermodynamics (magnetization, ferromagnetic transition) and stability (regular vs. chaotic behavior) of the model both analytically and numerically. Due to the nature of the method however (at every step of time we must keep account of the two replicas), there is obvious room for improving the computational efficiency. This is also the case in the analytical realm, where accounting for the way in which at each step of time the differences between the two copies may increase inevitably leads to lengthy computations (as was shown in [2, 3, 4, 5, 6]). This on itself constitutes a second motivation in order to search for alternative methods.

The method of analysis that we will propose here, so-called the self-overlap method (SO) [18, 19, 20], has already been successfully used in the study of the stability and critical points of random Boolean networks, a system that is multi-component albeit deterministic. In this work we will show that the method can also be successfully applied to a stochastic system such as a spin network. We will obtain analytical and numerical results on the $2D$ Glauber-Ising model that exactly match those yielded by DS. However, contrary to DS, SO proceeds by handling only one replica and analyzing its own evolution in time –using basically one-point correlation functions at subsequent time steps for that task. The computational costs are thus lower in SO than in DS. As we will see the analytical calculations also become much simpler while yielding the same results. Furthermore, and what is more important, the SO method is free from the ambiguities that afflicted DS due to its use of two replicas. This comes as a direct consequence of the already mentioned fact that on a single replica it does not matter whether one uses Glauber or HB dynamics since they posses the same one-point correlation functions.

We will follow the development applied by Vojta in [2], comparing in each case the results obtained using DS and our results (using SO). The paper is organized as it follows: in section II we quickly introduce both the $2D$ Glauber-Ising model and SO. We then apply the method to the $2D$ Glauber-Ising model in section III, obtaining a system of equations (master equation) that describe the dynamical evolution of the system. We discuss then how to apply a mean-field approximation to the system, and compare it with the methodology used by Vojta [2]. In section IV we obtain an analytical expression for the magnetization of the system in both ferromagnetic/paramagnetic phases similar to that obtained by Vojta [2]. Numerical results are provided at this point in order to validate the mean field approximation assumed in the analytical development. Finally, in section V we provide some analytical and numerical results on the stability of the model, showing that the system is chaotic (disordered) in the paramagnetic phase. Conclusions are presented in section VI.

II. ISING MODEL, THE DAMAGE SPREADING VS. SELF-OVERLAP METHOD

A. Glauber Ising model

We will work with a kinetic Ising model, a lattice of N spins, $s_i \in \{+1, -1\}$, that follows Glauber dynamics. That is, at every time step a lattice site i is chosen at random. If the spin value of site i at time t is given by $s_i(t)$, at time $t+1$ it will be given by:

$$s_i(t+1) = \operatorname{sgn}\left[\phi(h_i(t)) - \frac{1}{2} + s_i(t)\left(\xi_i(t) - \frac{1}{2}\right)\right], \tag{1}$$

where $\xi_i(t)$ is a random number such that $\xi_i(t) \in [0, 1)$. The transition probability $\phi(h_i)$ is given by the usual Glauber expression:

$$\phi(h_i(t)) = \frac{e^{h_i(t)/T}}{e^{h_i(t)/T} + e^{-h_i(t)/T}} \tag{2}$$

where T denotes the temperature and $h_i(t)$ is the local field seen by spin i at time t:

$$h_i(t) = \sum_{j=n.n.} J_{ij} s_j(t) + h_0. \tag{3}$$

In this expression h_0 represents an external magnetic field, and the sum in the interaction term applies only to the nearest neighbors (three for example in an hexagonal lattice). Without loss of generality from now on we will take $h_0 = 0$ and $J_{ij} = 1$.

B. Damage spreading and self-overlap

As stated above we will use the SO method to study the dynamics of the system. This procedure was introduced by Luque and Ferrera [18] and its underlying philosophy is similar to that of the DS method used by Vojta to study the thermodynamics of phase transitions in spin systems. The main difference between the two procedures lies in that, while damage spreading uses two

copies of a system with slightly different initial conditions (the *damage*) and computes the evolution of these differences, the self-overlap method uses the difference between successive temporal states of a single system *as the system evolves towards equilibrium*. For instance, in DS the damage $D(t)$ at time t is defined as:

$$D(t) = \frac{1}{2N}\sum_{i=1}^{N}\left| s_i^{(1)}(t) - s_i^{(2)}(t)\right| \tag{4}$$

and measures the (averaged) Hamming distance between the states of the two replicas at that time (i.e., the proportion of sites for which the spin state differs between the system (1) and the damaged replica (2)). In SO however the self-overlap $a(t)$ at time t is defined as one minus the averaged Hamming distance between the states of a spin site at time $t-1$ and at time t:

$$a(t) = 1 - \frac{1}{2N}\sum_{i=1}^{N}\left| s_i(t) - s_i(t-1)\right|. \tag{5}$$

In order to describe the time evolution of the system it is useful to define the "up state self-overlap" $a_{++}(t)$ at time t as the average number of spin sites that had $s_i = +1$ both at time $t-1$ and at time t. We also define $a_{--}(t)$, $a_{+-}(t)$ and $a_{-+}(t)$ in a completely similar fashion. By normalization we must then have:

$$a_{++}(t) + a_{--}(t) + a_{+-}(t) + a_{-+}(t) = 1. \tag{6}$$

Since the sites that remain in the same state at times $t-1$ and t drop from the sum in the definition (5) we also must have

$$a(t) = 1 - a_{+-}(t) - a_{-+}(t) = a_{++}(t) + a_{--}(t). \tag{7}$$

Once the equilibrium as been reached the relation $a_{+-} = a_{-+}$ must be satisfied, where we have dropped the time dependence to indicate equilibrium values. Then trivially

$$a_{+-} = a_{-+} = \frac{1-a}{2}. \tag{8}$$

At this point it is interesting to note that the self-overlap functions can be understood in terms of autocorrelation functions, more precisely, two-time autocorrelation functions. For instance, in equation (5), one can rewrite $|s_i(t) - s_i(t-1)|$ as $(1 - s_i(t)s_i(t-1))/2$, which is a shifted autoresponse function measured at subsequent time steps. In a similar way, the rest of self-overlap functions can be written as linear combinations of the basis of autoresponse functions.

Autocorrelation functions have been widely used as efficient tools in order to measure spatial or temporal correlations in physical and biological systems (repeated patterns, relaxation, frustration, etc). Their applications range from investigations in transport properties of fluids [21] or the analysis of climatological models [22] to studies of decoherence in quantum systems [23], to cite but a few. Autocorrelation functions are the center of interest in theoretical studies of the relaxation of non-equilibrium systems. In this sense, much work has been recently done in order to characterize dynamical scaling and other invariant behavior in the ageing regimes of Ising-like systems [24, 25, 26, 27, 28]. In our case it would be fair to say that the self-overlap functions are really measurements of autocorrelations under a different garment. To dwell on a deeper review of the existing literature on autoresponse functions would go beyond the scope of this paper however. We would like to emphasize nonetheless that what is new here is: *i)* the fact that this particular combination of self-correlation functions measured at subsequent time steps manages to capture the essence of the (same-site) temporal correlations in systems that undergo order/disorder phase transitions, and *ii)* this is then combined with a philosophy inspired by DS, namely: an evolution equation towards the equilibrium state for the correlations, and a mean field approximation directly extracted from DS in order to be able to solve this equation. Once the evolution equation and the mean field approximation are in place the self-overlaps will allow us to study the stability of the different states accessible to the system, and hence the phase transition itself.

III. MASTER EQUATION, TRANSITION PROBABILITIES, AND MEAN FIELD

A. Master equation

Generally speaking, the self-overlap method would proceed by solving some master evolution equation for the a's in order to obtain their equilibrium values, much in the vein of the damage spread method. We begin by defining the probability of finding a spin site in the $+$ $(-)$ state at time t, $P_+(t)$ $(P_-(t))$

$$P_\pm(t) = \frac{n_\pm(t)}{N}, \tag{9}$$

where $n_\pm(t)$ is the number of sites with spin up (down) at time t. Obviously $P_+(t)+P_-(t) = 1$. By the definition of $a_{++}(t), a_{-+}(t)$ it follows that

$$P_+(t) = a_{-+}(t) + a_{++}(t) = \frac{1 + a_{++}(t) - a_{--}(t)}{2} \tag{10}$$

and analogously for the down states

$$P_-(t) = a_{+-}(t) + a_{--}(t) = \frac{1 + a_{--}(t) - a_{++}(t)}{2}. \tag{11}$$

As noted above in the limit $t \to \infty$ the a's ought to reach their equilibrium values and one can drop the t dependence.

Of particular interest to us will be the transition probabilities from one state to another, i.e., the elements of

the transition matrix of our Markov process. Let $W_{++}(t)$ ($W_{--}(t)$) be the *average* probability of changing from the + (−) state at time t to the + (−) state at time $t+1$, where the precise meaning of this average will be made clear shortly. In an mean field approximation we will then have

$$a_{++}(t) = W_{++}(t-1)P_+(t-1),$$

$$a_{--}(t) = W_{--}(t-1)P_-(t-1) \tag{12}$$

and analogously with W_{+-}, W_{-+}. Note that these W's will then be the elements of an average Markov matrix for the evolution of the system. Combining (10),(11) and (12) together it is easy to arrive to a couple of mean field evolution equations for $a_{++}(t)$ and $a_{--}(t)$, namely

$$\begin{aligned} \frac{d}{dt}a_{++}(t) &= -a_{++}(t)W_{+-}(t) + a_{-+}(t)W_{++}(t), \\ \frac{d}{dt}a_{--}(t) &= -a_{--}(t)W_{-+}(t) + a_{+-}(t)W_{--}(t). \end{aligned} \tag{13}$$

These two equations are of course nothing but the reaction-diffusion equations for the a's that common sense would have dictated us to begin with. We now proceed to evaluate a mean field approximation for the W's so that we may solve (13).

B. Mean-field approximation

To begin with, note that in a system that follows Glauber dynamics the transition probability at site i for a given local field h_i is given by (2) above. This means that

$$\begin{aligned} W_{++}(h_i) &= \phi(h_i), \qquad W_{+-}(h_i) = 1-\phi(h_i), \\ W_{-+}(h_i) &= \phi(h_i), \qquad W_{--}(h_i) = 1-\phi(h_i). \end{aligned} \tag{14}$$

That is, as is well known for a given local field h_i the probability that the spin at site i will be in the + state at time $t+1$ is always $\phi(h_i)$, whereas the probability that its state be − will be $1-\phi(h_i)$, regardless of the initial state of the site. Thus, finding average values for the W's is equivalent to finding an average $\phi(h_i)$, $\overline{\phi}$.

The mean field approximation that we will use closely follows the spirit of the effective-field approximation used by Vojta [2]. This consists basically in averaging over all the possible configurations that can surround a given site, where in the average each configuration is weighted by its probability of taking place. Thus, with three nearest neighbors per site, the transition probabilities can take the values (remember that we are taking $J_{ij}=1$)

$$\phi_0\{+++\} = \phi(3) = \frac{e^{3/T}}{2\cosh(3/T)},$$

$$\phi_1\{++-\} = \phi(1) = \frac{e^{1/T}}{2\cosh(1/T)},$$

$$\phi_2\{+--\} = \phi(-1) = \frac{e^{-1/T}}{2\cosh(-1/T)},$$

$$\phi_3\{---\} = \phi(-3) = \frac{e^{-3/T}}{2\cosh(-3/T)}. \tag{15}$$

Note that the calculations are much simpler than those needed in DS [2]. The probability associated to each configuration will be

$$P(\phi_0) = P_+^3,$$

$$P(\phi_1) = 3P_+^2(1-P_+),$$

$$P(\phi_2) = 3P_+(1-P_+)^2,$$

$$P(\phi_3) = (1-P_+)^3, \tag{16}$$

where to simplify the notation we have dropped the time dependence, although in this case one must be aware that we are not dealing with equilibrium values (this will be the case for the next several equations). Using equations (10) and (11), we can now write after some trivial manipulations

$$\overline{\phi} = \sum_{k=0}^{3} P(\phi_k)\phi_k =$$

$$\begin{aligned} &\frac{1}{2} + \frac{3}{8}(a_{++}-a_{--})\left[\tanh\left(\frac{3}{T}\right) + \tanh\left(\frac{1}{T}\right)\right] \\ &+\frac{1}{8}(a_{++}-a_{--})^3\left[\tanh\left(\frac{3}{T}\right) - 3\tanh\left(\frac{1}{T}\right)\right]. \end{aligned} \tag{17}$$

Using the relations between the a's and applying the mean field to the right hand side of the differential equations (13) we can rewrite them as

$$\frac{d}{dt}a_{++} = -a_{++}(1-\overline{\phi}) + \left(\frac{1-a_{++}-a_{--}}{2}\right)\overline{\phi} \tag{18}$$

$$\frac{d}{dt}a_{--} = -a_{--}\overline{\phi} + \left(\frac{1-a_{++}-a_{--}}{2}\right)(1-\overline{\phi}), \tag{19}$$

which by (17) is now a system of equations depending only on a_{++} and a_{--}. Note that it is easy to generalize the mean field approximation to the case of n nearest neighbors (that is, for a given topology):

$$\overline{\phi} = \sum_{k=0}^{n}\binom{n}{k}P_+^{n-k}P_-^k\frac{1}{1+\exp\left(\frac{2n-4k}{T}\right)}. \tag{20}$$

This would be much harder to do using DS, if at all possible.

IV. THERMODYNAMICS: MAGNETIZATION

At this point we are going to link the self-overlaps to the average magnetization per spin, m. With $P_+(t), P_-(t)$ as defined above

$$P_\pm(t) = \frac{n_\pm(t)}{N}, \tag{21}$$

we must then obviously have for the average magnetization m

$$m(t) = P_+(t) - P_-(t), \tag{22}$$

or, since $P_+(t) + P_-(t) = 1$,

$$P_+(t) = \frac{1+m(t)}{2}, \qquad P_-(t) = \frac{1-m(t)}{2}. \tag{23}$$

By the definition of $a_{++}(t), a_{-+}(t)$ it follows then

$$a_{-+}(t) + a_{++}(t) = P_+(t) = \frac{1+m(t)}{2}, \tag{24}$$

and analogously with $a_{--}(t), a_{+-}(t)$ and $P_-(t)$.
Since

$$m(t) = a_{++}(t) - a_{--}(t), \tag{25}$$

the system of equations (17,18,19) can be rewritten as

$$\frac{d}{dt}m = \frac{m}{2}\left\{-1 + \frac{3}{4}\left[\tanh\left(\frac{1}{T}\right) + \tanh\left(\frac{3}{T}\right)\right]\right\}$$

$$+ \frac{m^3}{8}\left\{\tanh\left(\frac{3}{T}\right) - 3\tanh\left(\frac{1}{T}\right)\right\}. \tag{26}$$

Within the limits of our approximation this equation describes the evolution towards equilibrium of the magnetization m for the case of $n = 3$ nearest neighbors. Setting $dm/dt = 0$ one can obtain an expression for the temperature dependence of its equilibrium value $m(T)$, and from it one can extract the transition temperature for the ferro-paramagnetic transition —this was the approach originally followed by Vojta [2].
Equation (26) yields a critical temperature $T_c \approx 2.104$ above which the magnetization is zero. When $T < T_c$, we have:

$$m = \pm\sqrt{\frac{-1 + \frac{3}{4}\left[\tanh\left(\frac{3}{T}\right) + \tanh\left(\frac{1}{T}\right)\right]}{\frac{3}{4}\tanh\left(\frac{1}{T}\right) - \frac{1}{4}\tanh\left(\frac{3}{T}\right)}}. \tag{27}$$

Both results completely coincides with those in [2]. Note however that the calculations involved here have been considerably simpler —again basically due to the fact that in SO we only consider one replica of the system, which results in a considerable reduction in the number of configurations that need to be taken into account.

In our Monte Carlo simulations, the procedure to measure the (equilibrium) self-overlap goes as follows: let us suppose that we generate a random initial condition for the N spin lattice. Then we let it evolve towards equilibrium by applying the Glauber dynamics with 4 neighbors (square lattice). Once equilibrium has been reached we compute the states of the system for a sufficiently large number of time steps. We have used in all cases $10,000 \times N$ time steps for a square lattice of $N = 100 \times 100$ spins (that is, defining a system time step t as N steps of the simulation, we use $t = 10000$ system time steps). If we then count the number of times that a spin site is in the "up" state, $+$, both at time t and $t-1$ and average over all sites and time steps, this will give us the equilibrium value of the up state self-overlap a_{++}. Repeating this procedure with the down state, $-$, will then obviously give us the "down state self-overlap", a_{--}, and so on. Each value of the simulation is averaged over 100 realizations.
In figure (1) we plot the average equilibrium magnetization vs. temperature in order to visualize how our mean-field approximation performs —we note here that we are basically interested in the thermodynamic limit of infinite lattice size and that we are removing the inherent degeneracy of the system by plotting only positive magnetization. First, note that our Monte Carlo simulations in a square lattice (squares) are in fair agreement with the Onsager (infinite size) solution –dashed line– except in the proximity of the phase transition, where finite size effects are relevant and difficult to suppress. Comparing then the Monte Carlo simulations and the mean-field solution with $n = 4$ neighbors we can see that qualitatively speaking they provide the same results, with the mean field typically overestimating the critical temperature. We stress here however that the purpose of this paper was not so much to present a mean field technique able to reproduce the exact results, but rather to introduce a new technique able to exactly reproduce previously known mean field results while at a much lower cost. For illustrative purposes and to allow comparison with the results obtained by Vojta we also show in figure 1 the mean-field result for $n = 3$ neighbors (hexagonal lattice), which underestimates the $n = 4$ critical temperature T_c.

In figure (2) we plot the equilibrium values a^*_{++}, a^*_{--} vs. temperature, following the same methodology of figure (1): we compare our Monte Carlo simulations (circles) with the numerical resolution of the mean field equations (note that again, the mean field with $n = 3$ underestimates the quantitative behavior and the one with $n = 4$ overestimates it). As we can see in the figure, the self-overlap $a = a_{++} + a_{--}$ acts as an order parameter.

We also note that more work remains to be done in order to make an in-depth comparison between DS and

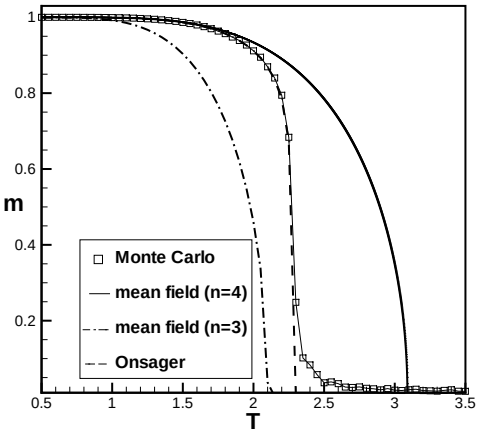

FIG. 1: Magnetization of the system versus temperature in the case of: (squares) Monte Carlo simulation of a 100×100 spin square lattice (the solid line here is just a guide for the eye), with $10,000$ system steps and averaged over 100 realizations ; (dashed-dot line) mean field approximation for $n = 3$ neighbors; (solid line) mean field approximation for $n = 4$ neighbors; (dashed line) Onsager solution. Note that the mean field approximation recovers the expected behavior, that is, null magnetization above T_c, non null magnetization below T_c, which tends to a constant maximum value at $T = 0$. The difference lies on the quantitative value of T_c in each case, overestimated by the mean field in the case of $n = 4$ neighbors.

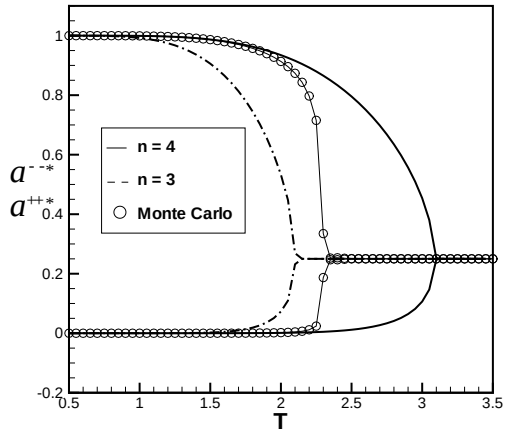

FIG. 2: Stationary values of a_{++} and a_{--} in the case of: mean-field approximation with $n = 3$ first neighbors (dashed line), mean-field approximation with $n = 4$ first neighbors (solid line) and Monte Carlo simulation of a 100×100 spin square lattice (here the solid line is just a guide for the eye), with $10,000$ system steps and averaged over 100 realizations (circles). Note that at T_c a pitchfork bifurcation takes place in the three cases. The bifurcation value is underestimated by the mean field approximation in the case of $n = 3$ neighbors and overestimated in the case of $n = 4$ neighbors. Below the critical temperature $a_{++} \to 1$ while $a_{--} \to 0$ for a system that chooses the $m = +1$ vacuum, whereas the opposite is true if the system goes to $m = -1$. Above T_c the system tends to $(a_{++},a_{--})=(1/4, 1/4)$. Note that although the critical temperature is only predicted qualitatively, the stationary values for (a_{++},a_{--}) yielded by our simple model exactly match the Onsager predictions.

SO. For instance, one may evaluate the critical exponents of the self-overlap order parameter a and compare the results with the DS approach [9], which would be interesting. This however goes somewhat beyond the scope of this paper.

V. STABILITY

In equation (26), the stability of the fixed point $m^* = 0$ (paramagnetic phase) is related to the sign of the eigenvalue:

$$\lambda(T) = 1/2(-1 + 3/4[\tanh(1/T) + \tanh(3/T)]). \quad (28)$$

Note that (26) falls into the normal form of a pitchfork bifurcation at $T = T_c$ where the fixed point is not hyperbolic and the Hartman-Grobman theorem [29] does not apply. For $T > T_c$, $m^* = 0$ is stable, and below it, it becomes unstable. The fact that we have a pitchfork bifurcation at T_c implies that in the ferromagnetic phase (i.e., below T_c) two other stable fixed points must appear. They are indeed $\pm m^*$, where m^* is now given by (27). Taking into account the relation between m and a, with a little algebra we arrive at

$$a^* = m^*(1 - \overline{\phi}^*) + \overline{\phi}^*. \quad (29)$$

Hence, the fixed point $m^* = 0$ leads to $\overline{\phi}^* = 1/2$ (according to (17)) and $a^* = 1/2$, which are thus stable at $T > T_c$. Note that $a = 1/2$ is the minimal self-overlap that the system can show.

We can define at this point a Hamming-like distance between successive temporal states (a self-distance), as:

$$d(t) = 1 - a(t). \quad (30)$$

The fixed point $a^* = 1/2$ implies that we must have a fixed point for d at $d^* = 1/2$ which, since it is taking place at the minimal self-overlap, is equivalent to the maximal self-distance of the system (total disorder). Following Wolf's method as in the case of random Boolean networks [19], this self-distance would enable us to determine a Lyapunov exponent of the system. However, one can simply apply the Hartman-Grobman theorem directly [29]. Near the fixed points the self-distance of our system can be expressed in terms of $d(t) \sim \exp(\lambda t)$, where λ is given by (28) . This eigenvalue can also be understood as a Lyapunov exponent. Note that nevertheless it would not be a standard Lyapunov exponent: when $d(t)$ tends to its fixed point, the system is actually tending to the maximal disorder, thus $\lambda < 0$ means chaos.

Summing up, in the paramagnetic phase, $m^* = 0$ is

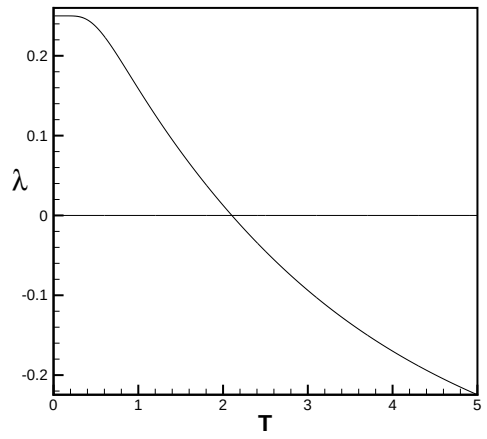

FIG. 3: Values of the temperature dependent eigenvalue (28) of J: when it is negative, $(a_{++},a_{--})=(1/4,1/4)$ is stable, thus the self-distance $d = 1/2$ is the attractor of the system (chaotic phase). When the eigenvalue is positive, the value $(1/4,1/4)$ is a saddle point and thus an unstable fixed point. At $T_c \approx 2.104$ the eigenvalue is null, thus the fixed point is not hyperbolic -a bifurcation takes place-.

stable, thus $d^* = 1/2$ is stable too: the system tends exponentially to the maximal disorder and the phase is chaotic.
Figure (3) is a plot of equation (28). Note that when $T > T_c$ (paramagnetic phase) an increase of the temperature leads to an increase of chaos, with the self-distance of the system tending faster to the attractor $d^* = 1/2$.

In the ferromagnetic phase however the stable stationary value of d is

$$d^*(T) = 1 - m^*(T)(1 - \overline{\phi}^*) - \overline{\phi}^*, \qquad (31)$$

with m^* given by (27) and $\overline{\phi}^*$ the fixed point value of the mean field. The self-distance tends to zero for low T, and thus the system is in a frozen state (order). When we increase the temperature the self-distance also increases up to the maximum value $d = 1/2$, which is reached at T_c (no correlation). These results agree with those found in the paramagnetic phase. We can conclude therefore that our approach correctly reproduces an ordered behavior in the ferromagnetic phase and disordered (chaotic) behavior in the paramagnetic phase. In Appendix (A) we perform a more detailed analysis of the stability of the system that confirms this conclusion.

VI. CONCLUSION

In this paper we have introduced the self-overlap method by using it to study both analytically and numerically the $2D$ Ising model. Since the properties of this model are obviously well known our main concern was to show that SO is an unambiguous method (with respect to changes in the algorithm implementation) that correctly reproduces the standard results while being very advantageous from both the numerical and the analytical point of view. The SO method could thus constitute a rather simple and efficient method of stability analysis in this kind of multicomponent systems (Ising-like models, spin glasses, CA, Kauffman networks, etc). Many other physically relevant quantities in these systems (measures of complexity, information theory measures such as the mutual information, and so on) can be studied and measured by applying SO, something that we think deserves further investigation. Wherever damage spreading was supposed to have been useful and the equilibrium state of the system is ergodic, we think that self-overlap ought to work too and do so in a non ambiguous manner. Moreover, it should also be more efficient numerically speaking, and simpler from the analytical viewpoint.

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APPENDIX A: DETAILED ANALYSIS OF THE STABILITY

We undertake here a deeper study on the stability of the system. For this task we go back to the evolution equations (18, 19), which constitute a nonlinear differential system. The fixed points of this system are obtained from equating (18,19) to zero (reducing the differential system to a linear system). This yields a total of three fixed points, namely:

$$(a_{++}(T)^*, a_{--}(T)^*) = \left(\frac{\overline{\phi}}{2}(1+m^*), m^*(\frac{\overline{\phi}}{2}-1)+\frac{\overline{\phi}}{2}\right),$$
$$(a_{++}(T)^*, a_{--}(T)^*) = \left(m^*(\frac{\overline{\phi}}{2}-1)+\frac{\overline{\phi}}{2}, \frac{\overline{\phi}}{2}(1+m^*)\right) \qquad (A1)$$

when $T < T_c$ (where m^* is given by (27)), and $(a_{++}(T)^*, a_{--}(T)^*) = (1/4,1/4)\ \forall T$ (this solution is obviously related to the fixed point $m^* = 0$).
We can write $\overline{\phi}$ as

$$\overline{\phi} = \frac{1}{2} + \frac{1}{2}A(T)(a_{++} - a_{--}) + \frac{1}{8}B(T)(a_{++} - a_{--})^3, \quad (A2)$$

where

$$A(T) = \frac{3}{4}[\tanh(3/T) + \tanh(1/T)], \qquad (A3)$$

and

$$B(T) = [\tanh(3/T) - 3\tanh(1/T)]. \qquad (A4)$$

Let's start with the stability analysis of the fixed point $(a^*_{++}, a^*_{--}) = (1/4, 1/4)$. This solution is independent of T and for $T > T_c$ is the only fixed point (note that in this case $\overline{\phi}$ takes the value $1/2$ independently of the number n of neighbors as it can be proved after some trivial algebra) . Computing the jacobian J at this fixed point, we come to:

$$J\,|_{(1/4,1/4)=\frac{1}{4}}\begin{pmatrix} A(T)-3 & -A(T)-1 \\ A(T)-1 & -A(T)-3 \end{pmatrix},$$

with eigenvalues $\lambda_1 = -1$, and $\lambda_2 = 1/2(A(T)-1)$. We will distinguish then three situations: when $A(T) < 1$, $(1/4, 1/4)$ is an hyperbolic (indeed stable) fixed point (which is obviously related to the fact that $m^* = 0$ is stable when $T > T_c$). When $A(T) > 1$ the fixed point is again hyperbolic, but now it is unstable (a saddle point). In these two situations we can apply the developed formalism, due to the Hartman-Grobman theorem [29]. Hence, $A(T) < 1 \Leftrightarrow T > 2/\ln(2^{2/3}+1) \approx 2.104$ (and viceversa for $A(T) > 1$).

We thus get that when $T > T_c$ (that is, in the paramagnetic phase), the stationary solution $(1/4, 1/4)$ is stable. In the ferromagnetic phase however ($T < T_c$) this fixed point becomes unstable.

At this point we can introduce the self-distance defined in (30). The stability of the $(1/4, 1/4)$ solution directly implies that d will have a stable value of $1/2$ in the paramagnetic phase, whilst this value will become unstable in the ferromagnetic phase. Since in the paramagnetic phase $(1/4, 1/4)$ is the only fixed point the self-distance necessarily goes to the attractor (stable fixed point) $d^* = 1/2$, indeed exponentially due to the Hartman-Grobman theorem, and the phase is thus chaotic. However in the ferromagnetic phase $(1/4, 1/4)$ is unstable: orbits with initial conditions arbitrarily close from this fixed point will separate from it exponentially, correlations will take place and the phase will become ordered.

When $A(T) = 1$, applying Peixoto's theorem [29], we can conclude that $(1/4, 1/4)$ is a bifurcation point (lack of structural stability), that is, T_c constitutes a bifurcation value. What kind of bifurcation is taking place?. It is easy to see that the linearized system has a symmetry of the type $a_{++} - a_{--}$. Using this symmetry, the system of equations (18,19) can be transformed into (26). This equation falls into the normal form of a codimension one bifurcation, a pitchfork bifurcation (indeed, subcritical). This means that two branches of equilibria appear for $T < T_c$ associated with values of $m \neq 0$, either positive (positive branch) or negative. Undoing the change of variables we get that below T_c we must have, for a given T, two extra stationary points –other than $(1/4, 1/4)$– of the shape $[(a, b),(b, a)]$. These fixed points correspond obviously to (A1). Moreover, since as the Poincaré index is a topological invariant these two new fixed points are both stable in the ferromagnetic phase (in the paramagnetic phase the global index is $+1$ because the fixed point $(1/4, 1/4)$ is a sink, whereas in the ferromagnetic phase $(1/4, 1/4)$ is a saddle point with index -1, so the other two fixed points must have index $+1$). Depending on the initial conditions, the system will evolve to a fixed point of the shape (a, b) or to (b, a). In other words, the Ising model will give us either positive or negative magnetization in the ferromagnetic phase, depending on the initial condition. If the system starts at $T > T_c$, where the magnetization is zero, and we lower its temperature below the critical one, fluctuations will take the system either to the upper or to the lower branch indistinctively.

In figure (2) we plot together the stationary values (a^*_{++}, a^*_{--}) of the differential system (18,19) for both $n = 3$ and $n = 4$ nearest neighbors and the results from our Monte-Carlo simulation (again, a square lattice of 100×100 spins, where we ran $10,000$ system steps after reaching equilibrium, and averaging over 100 realizations). We can see that the results are qualitatively similar, that is, the stationary value $(1/4, 1/4)$ is stable above the Curie temperature and unstable below it. As expected, at T_c a pitchfork bifurcation takes place and when $T < T_c$ the system has two stable fixed points, i.e. (a, b) and (b, a) for each T.

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